Cor 6.6: If R, S are central simple k-algebras, then $R \otimes_k S$ is a central simple k-algebra

Proof. $Z(R \otimes S) = k$ by T6.3, $R \otimes S$ simple by T6.4.

Cor 6.7 If R is a fin. dim. c.s.a, $\dim_k R = n$, then $R \otimes R^{op} \cong End(R_k) \cong M_n(k)$.

Proof. R, R^{op} csa. $\rightarrow R \otimes R^{op}$ is a csa of dimension n^2.

${}_R R_R$ bimodule $\Rightarrow \exists$ k-algebra hom $\varphi: \begin{cases} R \otimes R^{op} \to End(R_k) \cong M_n(k) \\ r \otimes s \mapsto (x \mapsto rxs) \end{cases}$

$R \otimes R^{op}$ simple $\Rightarrow \varphi$ injective.

Comparing dimensions, φ is bijective. $\square$

Exm: $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}^{op} \cong M_4(\mathbb{R})$

Since also $\mathbb{H} \cong \mathbb{H}^{op}$, via $z = \alpha + \beta i + \gamma j + \delta k \mapsto \bar{z} = \alpha - \beta i - \gamma j - \beta k$

(since $\overline{zw} = \bar{w}\bar{z}$), in fact $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong M_4(\mathbb{R})$.

Thm 6.7' If R is a fin.-dim. central simple k-algebra, then

$[R : k]$ $(= \dim_k R)$ is a square.

Proof: $R = M_n(D)$ with D a division ring [T2.21]

$Z(D) \cong k \Rightarrow D$ is a finite dimensional division k-algebra, $m := [D : k]$

Let $\bar{k}$ be an algebraic closure of k

$\Rightarrow D \otimes_k \bar{k}$ has $\dim_{\bar{k}} (D \otimes_k \bar{k}) = m$ and is a central simple

$\bar{k}$-algebra [T6.3, T6.4] $\underset{[L2.3]}{\xrightarrow{\bar{k} \text{ alg. closed}}}$ $\underbrace{D \otimes_k \bar{k}}_{\text{dim. } m} \cong \underbrace{M_r(\bar{k})}_{\dim r^2} \Rightarrow m = r^2$.

$\Longrightarrow \dim_k A = n^2 \cdot \dim_k D = m^2 r^2$ $\square$

In particular: f.d. div. algebras $/k$ have square dimension!

<u>Def:</u> R fin. dim. central simple k-algebra $(R \cong M_n(D),\ D$ div. ring$)$

$\deg_k R := \sqrt{[R:k]}$ is the degree of R

$\operatorname{ind}_k R := \sqrt{[D:k]}$ is the index of R

<u>Note</u> $[R:k] = n^2 [D:k]$, so $\operatorname{ind}_k(R) \mid \deg_k(R)$.

6.3 Extension of Scalars for Semisimple Algebras

K field

<u>Exm:</u> $K = \mathbb{F}_p(t)$, $x^p - t$ is irreducible $/K$, but

not separable: in $K(\alpha) \cong \mathbb{F}_p(t)[y]/(y^p - t)$ $x^p - t = x^p - \alpha^p = (x-\alpha)^p$

has a root w. multiplicity p.

$\Longrightarrow K(\alpha)$ semisimple K-algebra, but

$$K(\alpha) \otimes_K K(\alpha) \cong K(\alpha)[y]/(y^p - \alpha^p) \cong \underbrace{K(\alpha)[y]/(y-\alpha)^p}_{\text{Jacobson radical } (y-\alpha)+(y-\alpha)^p}$$ is not semisimple

<u>Recall:</u> (1) An algebraic field extension L/K is separable if the minimal polynomial $m_\alpha \in K[x]$ of each $\alpha \in L$ is separable

$(:\Leftrightarrow m_\alpha$ has no repeated roots in $\overline{K} \Leftrightarrow m_\alpha' \neq 0)$

(2) If K has characteristic 0 or is finite, L/K is always separable

(3) (Primitive Element Theorem) If L/K is finite separable, there exists $\alpha \in L/K$ s.t. $L = K(\alpha)$.

<u>Theorem 6.8</u> Let L/K be a finite field ext.

Then $(\forall K'/K: \; L_{K'} := L \otimes_K K' \text{ semisimple}) \Leftrightarrow L/K$ separable.

<u>Proof:</u> „$\Leftarrow$" Let $L = K(\alpha)$, so $L = K[x]/(m_\alpha) = \langle 1, \alpha, \alpha^2, \dots, \alpha^{n-1} \rangle_K$

with $n = [L:K]$, and $m_\alpha \in K[x]$ separable.

$\Rightarrow L_{K'}$ has K'-basis $(1 \otimes 1, \alpha \otimes 1, \dots, \alpha^{n-1} \otimes 1)$,

and $\alpha \otimes 1$ satisfies m_α, so $L_{K'} \cong K'[x]/(m_\alpha)$

In $K'[x]$, $m_\alpha = p_1 \cdots p_r$ with monic irreducible $p_i \in K'[x]$.

m_α separable $\Rightarrow p_1, \dots, p_r$ are pairwise distinct prime elements of the PID $K'[x]$ (so comaximal, i.e., $(p_i) + (p_j) = K'[x]$ for $i \neq j$)

Chinese Remainder Theorem $\Rightarrow K'[x]/(m_\alpha) = K'[x]/(p_1) \times \dots \times K'[x]/(p_r)$

with each $K'[x]/(p_i)$ a field.

„$\Rightarrow$" Let $\alpha \in L$ be not separable, i.e., $m_\alpha \in K[x]$ not separable,

So, $m_\alpha = (x - \alpha_1)^{e_1} \cdots (x_r - \alpha_r)^{e_r} \in \overline{K}[x]$ with some $e_i \geq 2$,

$\alpha_1, \dots, \alpha_r$ pairwise distinct.

$$\Rightarrow \overline{K} \otimes_K K(\alpha) \cong \overline{K}[x]/(m_\alpha) \cong \overline{K}[x]/(x-\alpha_1)^{e_1} \times \dots \times \overline{K}[x]/(x-\alpha_r)^{e_r} \quad \text{(CRT)}$$

If $e_i \geq 2$, for some i, $\overline{K} \otimes_K K(\alpha)$ has nonzero nilpotent elements,

then so does $L \otimes_K \overline{K} \supseteq K(\alpha) \otimes_K \overline{K}$. Then $L_{\overline{K}}$ is not semisimple (a commutative semisimple ring has no nonzero nilpotents, because it is a product of fields).

<u>Def:</u> K field, R f.d. semisimple K-algebra.

Then $Z(R) = K_1 \times \dots \times K_r$, K_i/K finite field extensions.

(using Thm 2.18)

R is separable if each K_i/K is separable.

<u>Cor 6.9</u> (1) If R is a separable f.d. semisimple K-algebra, then $R \otimes_K K'$ is semisimple for all fields $K' \supseteq K$

(2) If R, S are f.d. semisimple K-algebras, and at least one is separable, then $R \otimes_K S$ is semisimple.

<u>Proof:</u> (1) $R \cong M_{n_1}(D_1) \times \dots \times M_{n_r}(D_r)$ with D_i/K f.d. div.-algebras, $Z(D_i) = K_i$ separable $/K$.

$$K' \otimes_K R \cong K' \otimes_K (M_{n_1}(D_1) \times \dots \times M_{n_r}(D_r))$$

$$\cong (K' \otimes_K M_{n_1}(D_1)) \times \dots \times (K' \otimes_K M_{n_r}(D_r))$$

Now $K' \otimes_K M_{n_i}(D_i) \cong K' \otimes_K (K_i \otimes_{K_i} M_{n_i}(D_i))$ $\quad (K_i = Z(D_i))$

$$\cong (K' \otimes_K K_i) \otimes_{K_i} M_{n_i}(D_i)$$

$$\overset{[T6.8]}{\cong} (L_1 \times \dots \times L_{s_i}) \otimes_{K_i} M_{n_i}(D_i) \qquad (L_i \text{ fields})$$

$$\cong (L_1 \otimes_{K_i} M_{n_i}(D_i)) \times \dots \times (L_{s_i} \otimes_{K_i} M_{n_i}(D_i))$$

and $L_j \otimes_{K_i} M_{n_i}(D_i)$ is simple by Thm 6.4 since $M_{n_i}(D_i)$ is central simple, L_i is simple.

(2) Reducing over finite products (as in (1)), wlog R, S are simple and $Z(S)/K$ is separable.

$$R \otimes_K S \cong R \otimes_K (Z(S) \otimes_{Z(S)} S) \cong (R \otimes_K Z(S)) \otimes_{Z(S)} S$$

Conclusion of (1) also holds if R not separable, but K'/K separable

$$\overset{(1)}{\cong} (R_1 \times \dots \times R_t) \otimes_{Z(S)} S = \prod_{i=1}^{t} \underbrace{(R_i \otimes_{Z(S)} S)}_{\text{simple by [T6.4].}}$$

R_i simple $Z(S)$-algebras

<u>Exm:</u> $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is semisimple, so $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ as $\mathbb{R}$-algebras.

<u>Exc:</u> Show $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \times \mathbb{C}$ via $z \otimes w \mapsto (zw, z\bar{w})$.

6.4 The Skolem–Noether Theorem

If R is a K-algebra, $u \in R^\times$, then $\alpha: R \to R$, $x \mapsto uxu^{-1}$ is a K-automorphism. Such automorphisms are called *inner automorphisms*.

<u>Thm 6.10</u> (*Skolem–Noether*) Let R be a fin.-dim. central simple K-algebra and S a simple K-algebra. If $f, g: S \to R$ are K-algebra hom. (necessarily injective), then there is an inner automorphism $\alpha: R \to R$ s.t. $\alpha \circ f = g$.

<u>Equivalently:</u> If $R_1, R_2 \subseteq R$ are isomorphic simple subalgebras of R, and $h: R_1 \to R_2$ is a K-alg. hom, there exists $u \in R^\times$ s.t.

$\forall x \in R_1$: $h(x) = uxu^{-1}$.

[Equivalence: "$\Rightarrow$" Take $f: R_1 \hookrightarrow R$, $g: R_1 \xrightarrow{h} R_2 \hookrightarrow R$

$\Rightarrow \exists u \in R^\times$: $\underline{uxu^{-1}} = uf(x)u^{-1} = g(x) = \underline{h(x)}$ $\forall x \in R_1$.

"$\Leftarrow$": $R_1 := f(S) \xrightarrow[\sim]{g \circ f^{-1}} R_2 := g(S)$ $\Rightarrow$ $g \circ f^{-1} = \alpha$, α inner $\Rightarrow$ $g = \alpha \circ f$.]

Cor 6.11: If R is a f.d. central simple K-algebra and $\varphi \in \mathrm{Aut}_K(A)$, then $\varphi(x) = uxu^{-1}$ for some $u \in R^\times$.

Lemma 6.12 R f.d. simple K-algebra, $M, N \in \mathrm{Mod}\text{-}R$, f.d. $/K$.

Then $M \cong N \Leftrightarrow \dim_K M = \dim_K N$.

Proof: "$\Rightarrow$" ✓ "$\Leftarrow$" as simple ordinion ring, R has a unique simple module U_R. $\Rightarrow$ $M \cong U_R^r$, $N \cong U_R^t$ $\Rightarrow$

$r \cdot \dim_K U = \dim_K M = \dim_K N = t \dim_K U \Rightarrow r = t \Rightarrow M \cong N$. □

Proof of T6.10: ${}_R R_R$ is an (R,R)-bimodule ($r(xr') = (rx)r'$)

Now $f, g: S \to R$ define $\underline{\text{two}}$ (S,R)-bimodule structures on R:

$s \bullet r = f(s)r$, $s \square r := g(s)r$ $(\forall r \in R, s \in S)$

Equivalently, these are $S^{op} \otimes_K R$ right module structures, with the same K-dimension. $S^{op} \otimes_K R$ is a f.d. simple K-algebra [T6.4].

so the bimodule structures are isomorphic!

Meaning: $\exists$ K-linear bijective $h: R \longrightarrow R$ s.t.

(i) $\forall s \in S \, \forall x \in R$: $h(f(s)x) = g(s)h(x)$

(ii) $\forall x, r \in R$: $\quad h(xr) = h(x)r$

$\overset{(ii),\, x=1}{\Longrightarrow}$ $h(r) = h(1)\cdot r \quad \forall r \in R$.

h bijective $\Longrightarrow$ $h(1) \in R^\times$, $\quad u := h(1)$

$\overset{(i),\, x=1}{\Longrightarrow}$ $\underbrace{h(1)}_{=u} f(s) = g(s) \underbrace{h(1)}_{=u} \Rightarrow g(s) = u f(s) u^{-1}$. $\square$

Applications: Shorter proofs of Wedderburn's Little Thm, ...

6.5 The Centralizer Theorem

If R is a K-algebra, $X \subseteq R$, then $Z_R(X) = \{r \in R : \forall x \in X : rx = xr\}$ is a subalgebra. Note: $X \subseteq Z_R(Z_R(X))$ „double centralizer".

Thm 6.13 (Centralizer Theorem) Let R be a fin.-dim. central simple K-algebra (K a field), $S \subseteq R$ a simple subalgebra.

(1) $Z_R(S)$ is simple

(2) $[Z_R(S):K][S:K] = [R:K]$

(3) $Z_R(Z_R(S)) = S$. (Double Centralizer Theorem)

Proof: $R \cong M_n(D) \cong \mathrm{End}(V_D)$ with ${}_RV$ the unique simple R-module, $D = \mathrm{End}({}_RV)^{op}$ [left version of T2.21 or T4.7]

${}_RV_D$ is an (R,D)-bimodule, hence (S,D) bimodule, i.e. left $S \otimes_K D^{op}$-module.

Claim: $Z_R(S) \cong \mathrm{End}({}_SV_D) = \mathrm{End}({}_{S\otimes D^{op}}V)$ (bimodule end.)

$$\Big[\ Z_R(S) \hookrightarrow R \xrightarrow{\sim} End(V_D)$$

$$r \longmapsto \mu_r\ ,\quad \mu_r(v) = rv$$

Show: $\{\mu_r : r \in Z_R(S)\} = End({}_S V_D)$

"$\subseteq$": let $r \in Z_R(S)$: $\quad \mu_r(sv) = rsv = srv = s\mu_r(v) \quad (s \in S, v \in V)$

"$\supseteq$": let $r \in R$ s.t. $\mu_r(sv) = s\mu_r(v)$ $\forall s \in S, v \in V$

$$\Rightarrow (rs - sr)v = 0 \quad \forall v \overset{{}_R V \text{ faithful}}{\Longrightarrow} rs - sr = 0. \qquad \Big]$$

$S \otimes_K D^{op}$ is simple [T6.4]

$\Longrightarrow S \otimes_K D^{op} \cong M_m(E) \cong End(W_E)$ with $E = End({}_{S \otimes_K D^{op}} W)^{op}$, ⊛ (E: div. ring.)

W_E unique simple $S \otimes_K D^{op}$-module

$\Rightarrow V \cong W^t$ as $S \otimes_K D^{op}$-modules, $t \geq 1$.

$$\Rightarrow End_{S \otimes D^{op}}(V) \cong End_{S \otimes D^{op}}(W^t) = M_t\big(\underbrace{End_{S \otimes D^{op}}(W)}_{E^{op}}\big)$$

$\Rightarrow Z_R(S) \cong M_t(E^{op})$ simple

(2) $[R:K] = n^2\,[D:K]$

$[S:K][D:K] = m^2\,[E:K]$ by ⊛

$[Z_R(S):K] = t^2\,[E:K]$

$n[D:K] = \dim_K V = \dim_K W^t = t \dim_K W = tm\,[E:K] \qquad (V \cong W^t)$

$$\Rightarrow [S:K][Z_R(S):K] = m^2 t^2 \frac{[E:K]^2}{[D:K]} = \cancel{m^2 t^2} \frac{[D:K]^{\cancel{2}} n^2}{\cancel{t^2 m^2 [D:K]}} = [R:K].$$

(3) $S \subseteq Z_R(Z_R(S))$

and by (2): $[R:K] = [Z_R(S):K][S:K]$

simple by T6.13(1)

$= [Z_R(Z_R(S)):K][Z_R(S):K)$

so $[S:K] = [Z_R(Z_R(S)):K] \Rightarrow S = Z_R(Z_R(S))$

<u>Cor 6.14</u> If $S \subseteq R$ are fin. dim. central simple algebras, then

$R \cong S \otimes Z_R(S)$ (as algebras)

<u>Proof:</u> S, $Z_R(S)$ commute in R

$\Rightarrow \exists K$-alg. hom $\varphi: S \otimes Z_R(S) \to R$, $s \otimes s' \mapsto ss'$ (UP)

$S \otimes Z_R(S)$ simple [T6.4] $\Rightarrow \varphi$ injective

$\xrightarrow[6.13(2)]{\text{dimensions}} \varphi$ surjective. $\square$